



KENNETH W. WACHTER

# Essential Demographic Methods

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**KENNETH W. WACHTER**

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*For Bernadette*



# Contents

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List of Figures and Tables . . . . .	xi
Preface . . . . .	xv
<b>Introduction: Why Study Demography?</b>	<b>1</b>
<b>1 Exponential Growth</b>	<b>5</b>
1.1 The Balancing Equation . . . . .	5
1.2 The Growth Rate $R$ . . . . .	9
1.3 The Exponential Curve . . . . .	14
1.4 Models and Parameters . . . . .	16
* 1.5 Taylor Series . . . . .	19
* 1.6 Logistic Growth . . . . .	22
1.7 Doubling Times . . . . .	24
<b>2 Periods and Cohorts</b>	<b>30</b>
2.1 Lexis Diagrams . . . . .	30
2.2 Period Person-Years Lived . . . . .	33
2.3 The Crude Rate Model . . . . .	35
2.4 The Infant Mortality Rate . . . . .	37
* 2.5 Person-Years and Areas . . . . .	39
2.6 Cohort Person-Years Lived . . . . .	41
2.7 The Stationary Population Identity . . . . .	44



<b>3 Cohort Mortality</b>	<b>48</b>
3.1 Cohort Survival by Analogy . . . . .	48
3.2 Probabilities of Dying . . . . .	52
3.3 Columns of the Cohort Life Table . . . . .	54
3.3.1 King Edward's Children . . . . .	54
3.3.2 From ${}_nL_x$ to $e_x$ . . . . .	59
3.3.3 The Radix . . . . .	61
* 3.4 Hazards and Survivors . . . . .	63
* 3.5 Gompertz Hazards . . . . .	65
3.6 Annuities and Insurance . . . . .	71
3.7 Mortality of the 1300s and 2000s . . . . .	74
<b>4 Cohort Fertility</b>	<b>79</b>
4.1 Generational Renewal . . . . .	79
4.2 Age-Specific Fertility . . . . .	83
4.3 ASFRs and the NRR . . . . .	84
4.4 Cohort Parity . . . . .	86
4.5 Natural Fertility . . . . .	89
<b>5 Population Projection</b>	<b>98</b>
5.1 Transition Matrices . . . . .	98
5.2 Structural Zeros . . . . .	101
5.3 The Leslie Matrix Subdiagonal . . . . .	103
* 5.4 The Leslie Matrix First Row . . . . .	106
5.5 Projecting Fillies, Mares, Seniors . . . . .	110
* 5.6 Multi-State Tables . . . . .	114
* 5.7 Population Renewal . . . . .	116
* 5.8 Variable $r$ and the Lexis Surface . . . . .	119
<b>6 Period Fertility</b>	<b>125</b>
6.1 Period Measures . . . . .	125
6.2 Period Age-Specific Fertility . . . . .	128
6.3 Period $NRR$ , $GRR$ , and $TFR$ . . . . .	130
* 6.4 $\text{Log}(GRR)$ Plots . . . . .	133
6.5 Age-Standardized Rates . . . . .	136
* 6.6 Tempo and Quantum . . . . .	138
6.7 Princeton Indices . . . . .	141
6.8 Coale and Trussell's $M$ and $m$ . . . . .	145
<b>7 Period Mortality</b>	<b>153</b>
7.1 Period Lifetables . . . . .	153

7.2 Gaps and Lags . . . . .	157
7.3 The 1660s and Laws of Mortality . . . . .	159
7.4 Graunt's Model Lifetable . . . . .	162
7.5 Coale–Demeny Model Lifetables . . . . .	164
7.6 Brass Relational Logit Models . . . . .	165
* 7.7 Lee–Carter Models . . . . .	169

## **8 Heterogeneous Risks 174**

8.1 Heterogeneity . . . . .	174
8.2 Multiple Decrements . . . . .	175
* 8.3 Competing Risks . . . . .	179
* 8.4 Calculations with Hazards . . . . .	181
* 8.5 Lifeluck, Risk, and Frailty . . . . .	185
* 8.6 Proportional Hazards . . . . .	187
* 8.7 Cox Regression Estimation . . . . .	191
* 8.8 Frailty Models . . . . .	194

## **9 Marriage and Family 201**

9.1 The Complexity of Marriage . . . . .	201
9.2 First Marriage by Analogy . . . . .	203
9.3 The <i>SMAFM</i> . . . . .	206
* 9.4 The Singulate Mean Formula . . . . .	209
9.5 Marity . . . . .	212

## **10 Stable Age Structures 218**

10.1 Age Pyramids . . . . .	218
10.2 Stationary Equivalent Populations . . . . .	221
10.3 Consequences of Unchanging Rates . . . . .	224
10.4 Stable Age Pyramids . . . . .	229
10.5 The Many Faces of Lotka's $r$ . . . . .	233
* 10.6 The Euler–Lotka Equation . . . . .	235
* 10.7 Life Left in Stable Populations . . . . .	241
10.8 Population Momentum . . . . .	243

## **11 Migration and Location 250**

11.1 Spatial Demography . . . . .	250
11.2 Flows of People . . . . .	251
11.3 Concentrations . . . . .	256
* 11.4 Random Walks . . . . .	262
11.5 GIS and Cartograms . . . . .	265

<b>Conclusion</b>	<b>271</b>
Appendix A: Sources and Notes . . . . .	273
Appendix B: Useful Formulas . . . . .	275
Bibliography . . . . .	279
Index . . . . .	285

# Figures and Tables

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## Figures

1.1	$K(t)$ with ever-changing slope . . . . .	10
1.2	$\log K(t)$ with constant slope . . . . .	10
1.3	Trajectories of exponential growth . . . . .	15
1.4	Rise and run: China's log-population . . . . .	15
2.1	A Lexis diagram . . . . .	31
2.2	Cohort, period, and age . . . . .	33
2.3	IMR contributions on a Lexis diagram . . . . .	38
2.4	Lexis diagram for a stationary population . . . . .	45
3.1	Lexis diagram for the children of King Edward III . . . . .	56
3.2	Lifespans and $\ell_x$ . . . . .	57
3.3	Typical shapes of lifetable functions . . . . .	61
3.4	An $\ell_x$ curve . . . . .	65
3.5	A Gompertz hazard function . . . . .	67
3.6	Logarithm of the hazard function . . . . .	69
4.1	Cohort fertility on a Lexis diagram . . . . .	80
5.1	Permitted transitions among marital statuses . . . . .	103
5.2	Contributions to the Leslie matrix subdiagonal . . . . .	104
5.3	Zigzag paths with Variable $r$ . . . . .	120
6.1	From period to cohort on a Lexis diagram . . . . .	127
6.2	Typical age-specific fertility schedules . . . . .	129
6.3	Log $GRR$ plot for England and Wales and for India . . . . .	135

6.4 Birth age standardization for tempo adjustment . . . . .	139
6.5 Fitting the Coale–Trussell model . . . . .	149
7.1 U.S. gaps and lags . . . . .	159
7.2 Shapes of Brass model lifetables . . . . .	167
8.1 Double jeopardy . . . . .	179
8.2 Cumulative hazard and median duration . . . . .	182
8.3 Partial likelihood . . . . .	194
9.1 The Singulate Mean as an area . . . . .	210
10.1 Examples of stable age pyramids . . . . .	219
10.2 Examples of observed age pyramids . . . . .	220
10.3 Projected growth rates over time . . . . .	225
10.4 An age group share over time . . . . .	226
10.5 Age group size over time . . . . .	227
10.6 Logarithm of Lotka’s integral for trial values of $r$ . . . . .	239
10.7 Logarithm of births in the Keyfitz scenario . . . . .	245
11.1 Rural and urban world populations . . . . .	252
11.2 Rank-Size Plot for U.S. cities 2010 . . . . .	257
11.3 Computing indices of separation . . . . .	261
11.4 Equal density cartogram for the United States . . . . .	267

## Tables

1.1 The world population 2010 to 2011 . . . . .	6
1.2 Growth rates in China . . . . .	16
1.3 Fitting Moore’s Law . . . . .	18
1.4 World populations and doubling times over history . . . . .	25
1.5 U.S. Census counts in millions . . . . .	28
2.1 The world’s 10 most populous countries . . . . .	37
3.1 An analogy between populations and cohorts . . . . .	49
3.2 Children of King Edward III of England . . . . .	55
3.3 Five columns of King Edward’s family lifetable . . . . .	58
3.4 Right-hand columns of a lifetable . . . . .	60
3.5 Cohort lifetable formulas . . . . .	62
3.6 Fitting Gompertz parameters . . . . .	70
3.7 Survivorship forecasts for a U.S. cohort . . . . .	77
3.8 Hazard rates for British women born in 1910 . . . . .	77

4.1	Generation sizes and the <i>NRR</i> . . . . .	81
4.2	A cohort <i>NRR</i> from U.S. age-specific rates . . . . .	85
4.3	Completed parity for U.S. women born in 1934 . . . . .	87
4.4	Dutch women age 50 by parity, 2009 . . . . .	88
4.5	Fertility rates specific for age and parity . . . . .	91
4.6	Data for the 1800 cohort of Swedish women . . . . .	97
4.7	Women by completed parity, Malawi, 2004 . . . . .	97
6.1	Age-specific rates for India, 2000 . . . . .	130
6.2	Calculating the <i>NRR</i> , India, 2000 . . . . .	131
6.3	An age-standardized birth rate . . . . .	137
6.4	Calculating $I_f$ and $I_g$ for Berlin in 1900 . . . . .	143
6.5	Coale–Trussell spine constants . . . . .	146
6.6	Coale–Trussell $M$ and $m$ for Bangladesh 2007 . . . . .	148
6.7	Period data for women in Togo for 1961 . . . . .	151
6.8	<i>GRR</i> and <i>TFR</i> from China . . . . .	152
6.9	Portuguese fertility 1930 and 1960 . . . . .	152
7.1	U.S. raw mortality data from 2010 . . . . .	156
7.2	The Graunt lifetable . . . . .	163
7.3	Spines for Brass general standard . . . . .	166
7.4	Estimation of Brass parameters . . . . .	168
8.1	Cause-specific death rates for Japan, ages 80–85 for 1981 to 1990 . . . . .	178
8.2	Lifeluck, risk, and frailty . . . . .	186
8.3	Proportional hazard parameters . . . . .	189
8.4	Record lifespans . . . . .	192
9.1	First marriages for the U.S. cohort of 1930 . . . . .	204
9.2	Marital breakdowns for Egypt in 1986 . . . . .	208
9.3	Marital status proportions for U.S. women 1990 . . . . .	217
10.1	Stable population data for India, youth, 2000 . . . . .	231
10.2	Stable population data for India, adults, 2000 . . . . .	231
11.1	The 10 most populous U.S. cities in 2010 . . . . .	257
11.2	Rural and urban populations in 2010 . . . . .	259



# Preface

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This book has grown out of courses in demographic methods that I have taught at the University of California, Berkeley, over the past thirty-five years. Students range from freshmen to pre-docs, and they bring a wide range of backgrounds to my courses. I have tried to make the book accessible to those new to quantitative analysis, but also to keep it interesting to those with prior training in mathematics and social sciences. Graduate-level material and any calculus are confined to sections marked with a star (\*).

One of the attractions of demographic methods is the opportunity to develop or refresh quantitative skills and overcome math anxiety while studying subjects of direct human and personal interest. For my courses there are no official prerequisites. Fluency with high-school mathematics is an advantage. The calculus used in the starred sections is explained along the way.

Studying demographic methods means, above all, drawing analogies and exploring examples. It thus provides opportunities to enjoy snippets from our cultural heritage along the way. In this book, the reader will come across a personal sampling of favorite passages and occasional excursions into metaphor.

I am indebted to the many students who have helped me develop this book, and especially to my outstanding Graduate Student Instructors Robert Chung, Debbie Blackwell, Bill Chu, Josh Goldstein, Laura Hill, Jie Huang, Aaron Gullickson, Bryan Sykes, Sarah Zureick, Maggie Frye,



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# **Essential Demographic Methods**



# Introduction

## Why Study Demography?

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As children, our earliest steps forward in awareness take us into demography, as we number our birthdays, put numbers on the ages of people around us, number the others in our family, our town, our country, and our planet, as we learn that the years we will have to live are numbered.

As adults, we cope with a world whose problems and opportunities are shaped by demography. Low growth rates and high growth rates form the backdrop to the contrasts between rich nations and poor nations, peace and conflict, environmental protection and degradation that form the stuff of each evening's news. The expanding proportion of elderly in developed societies challenges the social insurance and welfare systems. Reduced fertility is part and parcel of the transformation of gender roles reshaping social consciousness. Migratory movement puts strains on social cohesion. The transformations of marriage and family come to dominate political debate. Longer average lifespans give many of us new flexibility in planning out our life's fulfillments.

It is one thing to speculate freely about such trends, as all of us do. It is another to gain precise information about their extent so that definite numerical comparisons become possible. The science of demography has grown up over the last several centuries out of efforts to make measurements of population and lifecourse in consistent and meaningful ways. As such, it is an example, one of the best, of mathematics at work on the world.